Application of the parareal algorithm for acoustic wave propagation

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Abstract. We present an application of the parareal algorithm [1] to solve wave propagation problems in the time domain. The parareal algorithm is based on a decomposition of the integration time interval in time slices. It involves a serial prediction step based on a coarse approximation, and a correction step (computed in parallel) based on a fine approximation within each time slice. In our case, the spatial discretization is based on a spectral element approximation which allows flexible and accurate wave simulations in complex geological media. Fully explicit time advancing schemes are classically used for both coarse and fine solvers.

In a first stage, we solve the 1D acoustic wave equation in an homogeneous medium in order to test stability and convergence properties of the parareal algorithm. We confirmed the stability problems outlined by Bal [2] and Farhat et al. [3] for hyperbolic problems. These stability issues are mitigated by a time-discontinuous Galerkin discretization of the coarse solver. It may also involve a coarser spatial discretization (hp-refinement) which helps to preserve stability and allows more significant computer savings. Besides, we explore the contribution of elastodynamic homogenization to build consistent coarse grid solvers. Extension to 2D/3D realistic geological media is an ongoing work.

Keywords: Time parallelisation, acoustic waves, time-discontinuous Galerkin
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INTRODUCTION

The numerical simulation of seismic waves in complex geological structures is a significant problem in many fields such as exploration geophysics, hydrocarbon storage and prevention of seismic hazards. These problems require efficient solvers with low numerical dispersion for wave field propagation over a very large number of wavelengths.

The evolution of parallel computing architectures and new numerical methods (spectral elements method, high-order discontinuous Galerkin method) enabled dramatic advances in the field of 3D simulation of the full wave field in the areas of global and regional seismology [4, 5, 6]. These applications are already parallel in space by domain decomposition. The solvers in time are typically of the predictor-corrector type, second order in time.

In particular for the spectral elements method, it has been shown that in order to keep the spectral accuracy in long time simulations, the time step should be kept quite small, even much less than its stability limit determined by the ‘Courant-Friedrichs-Lewy’ (CFL) condition [7]. One motivation of this study is to evaluate the potential contribution of the parareal algorithm to obtain accurate solutions at large simulation times while reducing the computation time. For explicit methods, the challenge is to develop coarse rapid solvers that are not only based on larger time steps, but also on different space approximation and frequency content [8].

It is already known that the parareal algorithm in its classical formulation presents instabilities for hyperbolic problems [2, 3, 9] such as acoustic wave propagation and elastodynamics. In this work we face these stability issues by exploring more adequate time solvers such as the Time-Discontinuous Galerkin (TDG) method in the acoustic case. This choice is motivated by the inherent discontinuous problem to be solved by the coarse grid propagator in the parareal context.

PARAREAL ALGORITHM FOR THE ACOUSTIC WAVE EQUATION

We solve the 2nd-order homogeneous acoustic wave equation in a periodic 1D-domain. It can be expressed as a linear system of first order partial differential equations,

\[ \frac{\partial v}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} + f, \]  

(1)
where $u,v$ are the displacement and velocity field, $c$ is the acoustic wave velocity of the medium and $f$ an external source term. In the spectral element approach, the problem is solved using a variational (weak) formulation of equations (1) and (2). After spatial discretization using a Lagrangian polynomial basis based on Gauss-Lobatto-Legendre (GLL) points, we can write the semi-discrete spectral element system as,

\[ \frac{\partial u}{\partial t} = v, \quad \frac{\partial v}{\partial t} = F, \]  

where $M$ and $K$ are the assembled mass and stiffness matrices. Using a quadrature rule based on the GLL points, the mass matrix becomes diagonal by construction and the use of explicit time schemes becomes straightforward [4]. The parareal algorithm relies on the definition of a coarse (cheap) $G$ and a fine (expensive) $F$ solvers for the evolution problem expressed in (3) and (4). We refer to [2, 10] for a thorough explanation of the algorithm with emphasis in stability and convergence analysis.

The algorithm proposes an approximation of each snapshot of the fine solution $\lambda_n = [v_n, u_n]^T = F(0, T_n, \lambda_0)$, at times $T_n$, $n = 1,...,N$, by recursively defining the sequence $\lambda_n$ as

\[ \lambda_{n+1}^{k+1} = G(T_n, T_{n+1}, \lambda_n^{k+1}) + F(T_n, T_{n+1}, \lambda_n^k) - G(T_n, T_{n+1}, \lambda_n^k). \]

We divide the total simulation time in $N$ time slices and assign each slice $(T_n, T_{n+1})$ to a different processor. The method can be interpreted as a predictor-corrector scheme where the predictor phase $G(T_n, T_{n+1}, \lambda_n^{k+1})$ at iteration $k+1$ is sequentially calculated, while the correction term $F(T_n, T_{n+1}, \lambda_n^k) - G(T_n, T_{n+1}, \lambda_n^k)$ can be calculated in parallel for each processor (it only depends on results from iteration $k$).

Convergence towards the fine (sequential) solution is always obtained after enough number of iterations, in fact $\lambda_n^n = F(0, T_n, \lambda_0)$. However, in order to achieve significant speed-up, the convergence of $\lambda_n^k$ to $\lambda_n$ should go as fast as possible. Whereas the parareal algorithm rapidly converges for most discretizations of parabolic problems, it is unfortunately not the case for hyperbolic problems [2]. Finally, the coarse solver does not need to be as accurate as $F$ and can be chosen much less expensive, e.g. by the use of a scheme with a much larger time step or even with different spatial discretizations [8].

**Facing the instabilities: Time-Discontinuous Galerkin**

Two strategies have been presented to accelerate the convergence of the parareal algorithm for wave propagation and structural dynamics problems. Bal [11] proposed to subdivide the total time interval, and to apply the parareal algorithm at each subinterval where the initial condition is taken to be the $k = 1$ parareal solution from the preceding time interval. This strategy merits some more attention for wave propagation problems.

Another approach was proposed by Farhat and collaborators for linear [9] and non-linear [12] structural dynamics problems. They have shown that the inefficiency of the classical algorithm is caused by the so-called beating phenomenon [3], due to the propagation of jumps with the coarse propagator. The authors proposed to use the fine propagator $F$ for the part of the jumps already known from previous iterations, while propagate the rest with the coarse $G$ propagator. The main drawback is that the strategy needs to keep in memory the whole solution at every parareal iteration. This may not be feasible for 3D wave propagation problems with large (sometimes huge) number of degrees of freedom.

As it is clear from equation (5), the coarse solver is faced to discontinuities at each time slice where the correction term is added as a perturbation to the initial conditions at $T_n$. For this reason, we decide to explore time advancing schemes which incorporate jumps in a much more natural way than finite difference schemes, such as the time-discontinuous Galerkin method.

The method is based on a variational formulation in time of the semi-discrete problem of equations (3) and (4). At each time step, a residual equation in the interval $I_n = (t_n, t_{n+1})$ is constructed as

\[ \int_{t_n}^{t_{n+1}} w_v \cdot (M \dot{v} + Ku - F) dt + \int_{t_n}^{t_{n+1}} w_u \cdot (K \dot{u} - v) dt + w_{u_n} \cdot K[u_n] + w_{v_n} \cdot M[v_n] = 0, \]
FIGURE 1. Relative (max-norm) errors in the velocity field of the parareal/SEM solution versus time slices for the acoustic wave equation. (Left) explicit Newmark scheme for both coarse and grid solvers. (Right) TDG algorithm for the coarse solver from iteration 1 and higher, explicit Newmark for the fine solver.

where \([u_n] = u_n^+ - u_n^-\) represents the discontinuity (jump) of \(u\) at time \(t_n\), and \(w_u, w_v\) are the test functions for displacement and velocity fields, respectively. In the present work, following Wiberg & Li [13], we use 1st order polynomials in time for the test and the basis functions. Higher-order polynomial basis can be used [14] and we currently study the feasibility of their application. As in the classical Newmark approach, a fully explicit scheme can be easily derived in a predictor-corrector formulation [15].

NUMERICAL TESTS

We carry out some numerical simulations to test the TDG approach to the parareal/SEM algorithm for 1D acoustic wave propagation. An homogeneous 1D-medium of 5000 m length, 2000 m/s P-wave velocity and periodic boundary conditions is considered. We set as initial condition a Ricker source wavelet of central frequency \(f_0 = 2.5\) Hz centered at \(x = 2500\) m. The spatial discretization is based on 50 spectral elements of polynomial degree \(p = 6\), which assures more than 10 GLL nodes per minimum wavelength of the propagating wave. The total simulation time is 5 s, which corresponds to approximately 12 times the central source period. We use 50 time slices of \(\Delta T = 0.1\) s, with \(Dt = 4 \times 10^{-4}\) s (CFL = 0.68) and \(dt = 2 \times 10^{-5}\) (CFL = 0.034) for the coarse and fine time step, respectively. The errors to the analytical solution are easily calculated by time shifting the initial condition.

It is clear from Figure 1 that the parareal algorithm with an explicit Newmark scheme (centered differences) presents instabilities starting at 1 s (slice number 10) for iteration 2 and higher. When using the TDG algorithm for the coarse solver from the parareal iteration 1 on (iteration 0 corresponds to the coarse sequential solution), stability is recovered and convergence to the fine solution is achieved after iteration 2.

We have run exactly the same test with the TDG algorithm (similar \(Dt\) and \(dt\)) for both coarse and fine solvers. As expected for an \(O(\Delta t^3)\) solver, the errors to the analytical solution are lower than in the previous case (\(O(\Delta t^2)\)) as it is shown in Figure 2. It must be stressed that the total error of the coarse solver (with such a large CFL) is exclusively dominated by time errors. This is not the case for the fine sequential solution where spatial discretization errors still dominate. The parareal algorithm helps to recover that error level (\(\sim 10^{-6}\)) with a (theoretical) speed-up of 5 [2].

CONCLUSION AND CURRENT WORK

In this paper we explored the feasibility of the time-discontinuous Galerkin method in the parareal/SEM algorithm for acoustic wave propagation. It is clear from the previous simulations that the method offers more stable results of the parareal algorithm, while not degrading the mandatory rapid convergence to the fine sequential solution.

As it has been said, the coarse grid spatial discretization does not need to be exactly the same as the fine grid discretization. Indeed the coarse time step being much larger than the fine one, there is a need to tune the coarse spatial
discretization in order to recover stability (specially when using explicit time schemes). Secondly, the coarse solver should be as cheap as possible (i.e. less degrees of freedom) in order to maximize computing gains. The first numerical implementation of this reduction can be found in the work of Fischer and collaborators [8] for the Navier-Stokes problem. Two operators are needed in order to transfer the solution from the coarse into the fine grid (prolongation) and vice versa (restriction). In the spectral element context, they suggest the use of L2-operators which minimize the inner product of the residuals with the corresponding polynomial basis. Our tests with the parareal/SEM algorithm for the acoustic wave equation with different spatial discretizations do not present satisfactory convergence properties, unless a low-pass filter is applied at each time slice just after prolongation-restriction of the solution. Current work focuses on this interesting and important subject.

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REFERENCES